for this unit:

with polynomials

ratio $\Rightarrow$ fraction

same rules as regular fractions

Ex   Rational Expressions

$$\frac{5}{8} \quad , \quad \frac{3x}{4} \quad , \quad \frac{14z^2}{x-2} \quad , \quad \frac{18a^3 - 4b + 3c}{12a - c^8}$$

Restrictions

Denominator can't be zero

Ex   State the restriction(s), if any

a) $\dfrac{10x}{7}$ no restriction

for ex $\dfrac{0}{3} = 0$

b) $\dfrac{10}{7x}$ $x \neq 0$

$\dfrac{3}{0} \Rightarrow$ undefined

c) $\dfrac{x-2}{x-3}$ $x \neq 3$

d) $\dfrac{3x+2}{x^2-9} = \dfrac{3x+2}{(x-3)(x+3)}$ $x \neq -3, 3$

e) $\dfrac{3x+2}{x^2+9}$ ← Can't factor   no restriction

Simplifying Rational Expressions

Same rules
as fractions
$\Rightarrow$ $\dfrac{12}{18} = \dfrac{2 \cdot 2 \cdot 3}{2 \cdot 3 \cdot 3} = \dfrac{2}{3}$

12 → 4, 3 → 2, 2

18 → 2, 9 → 3, 3

Careful: $\dfrac{12}{18} = \dfrac{\cancel{10}+2}{\cancel{10}+8} \neq \dfrac{2}{8}$

① Factor top and bottom

② Cancel factors

Ex state restrictions then simplify

a) $\dfrac{4x+8}{x^2-4} = \dfrac{4 \cdot (x+2)}{(x-2)(x+2)}$ ← $x \neq -2, 2$

$= \boxed{\dfrac{4}{x-2}}$

b) $\dfrac{2x^3 - x^2 - 15x}{9x - x^3} = \dfrac{x(2x^2 - x - 15)}{x(9 - x^2)}$

$= \dfrac{-(x^3 - 9x)}{}$

$= \dfrac{\cancel{x}\,(x-3)(2x+5)}{\cancel{x}\,(3-x)(3+x)}$

$\Leftarrow$ $\begin{array}{c|cc} (\times) & -30 & (-6,5) \\ (+) & -1 & -1 \end{array}$

$2x^2 - 6x + 5x - 15$

$2x(x-3) + 5(x-3)$

$(x-3)(2x+5)$

note: $3 - x = -(x-3)$

$\boxed{x \neq 0, 3, -3}$

$= \dfrac{(\cancel{x-3})(2x+5)}{-(\cancel{x-3})(x+3)} = \boxed{-\dfrac{2x+5}{x+3}}$

Multiplying / Dividing

Same rules as fractions $\Rightarrow \dfrac{6}{7} \cdot \dfrac{14}{9} = \dfrac{2 \cdot \cancel{3}}{\cancel{7}} \cdot \dfrac{2 \cdot \cancel{7}}{\cancel{3} \cdot 3} = \dfrac{4}{3}$

① Factor top and bottom

② Cross reduce

③ multiply

Ex Simplify

a) $\dfrac{x^2-25}{x^2-4} \cdot \dfrac{x-2}{15-3x} = \dfrac{\cancel{(x-5)}(x+5)}{\cancel{(x-2)}(x+2)} \cdot \dfrac{\cancel{(x-2)}}{-3\cancel{(x-5)}}$

$15-3x = -(3x-15)$

$= \boxed{-\dfrac{x+5}{3(x+2)}}$

b) $\dfrac{x^2-3xy}{x^2-3xy-4y^2} \div \dfrac{3y^2+2xy-x^2}{x^2-5xy+4y^2}$

$\dfrac{6}{7} \div \dfrac{10}{4} = \dfrac{6}{7} \cdot \dfrac{4}{10}$

$= \dfrac{x^2-3xy}{x^2-3xy-4y^2} \cdot \dfrac{x^2-5xy+4y^2}{3y^2+2xy-x^2}$

$3y^2+2xy-x^2 = -(x^2-2xy-3y^2)$

$= \dfrac{x\,(x-3y)}{(x-4y)(x+y)} \cdot \dfrac{(x-y)(x-4y)}{-(x-3y)(x+y)}$

$\begin{array}{c|c} \otimes -3y^2 & (-3y,\ y) \\ \hline \oplus -2y & \end{array}$

$\begin{array}{c|c} \otimes -4y^2 & (-4y,\ 1y) \\ \hline \oplus -3y & -3y \end{array}$

$\begin{array}{c|c} \otimes\ 4y^2 & (-1y,\ -4y) \\ \hline \oplus -5y & -5y \end{array}$

$= \boxed{\dfrac{x\,(x-y)}{(x+y)^2}}$

Adding/Subtracting

Same as fractions $\Rightarrow$ $\dfrac{2}{3} + \dfrac{5}{3} = \dfrac{7}{3}$

$$\dfrac{2 \cdot 5}{3 \cdot 5} + \dfrac{3 \cdot 3}{5 \cdot 3} = \dfrac{10}{15} + \dfrac{9}{15} = \dfrac{19}{15}$$

① Make sure fractions have same denominator

② Add~~^{or subtract}~~ numerators, keep denominator the same

EX Simplify

a) $\dfrac{2x-1}{x^2-1} \ominus \dfrac{x-2}{x^2-1} = \dfrac{2x-1 \ominus (x-2)}{x^2-1} = \dfrac{2x-1-x+2}{x^2-1}$

$= \dfrac{x+1}{x^2-1} = \dfrac{(x+1)}{(x+1)(x-1)} = \boxed{\dfrac{1}{x-1}}$

$\dfrac{x-3}{x} \quad \cancel{\dfrac{x}{}} \quad \dfrac{-3}{1}$

b) $\dfrac{11x \cdot 3x}{6x^2 \cdot 3x} - \dfrac{7 \cdot 2}{9x^3 \cdot 2} = \dfrac{33x^2}{18x^3} - \dfrac{14}{18x^3} = \boxed{\dfrac{33x^2-14}{18x^3}}$

c) $\dfrac{1}{1-x^2} + \dfrac{3}{x^2+x} = -\dfrac{1}{(x-1)(x+1)}\cdot\dfrac{\cdot x}{\cdot x} + \dfrac{3}{x(x+1)}\cdot\dfrac{\cdot(x-1)}{\cdot(x-1)}$

$\underbrace{}$
$= -(x^2-1)$

$= \dfrac{-x}{x(x-1)(x+1)} + \dfrac{3(x-1)}{x(x-1)(x+1)} = \dfrac{-x+3x-3}{x(x-1)(x+1)}$

$= \boxed{\dfrac{2x-3}{x(x-1)(x+1)}}$

d) $\dfrac{14}{x^2+3x-10} + \dfrac{x+3}{x^2+3x+2} \cdot \dfrac{3x+3}{x+5}$

$= \dfrac{14}{x^2+3x-10} + \dfrac{x+3}{\cancel{(x+1)}(x+2)} \cdot \dfrac{3\cancel{(x+1)}}{x+5}$

$\begin{array}{c|c} \otimes \ 2 & (1,2) \\ \hline \oplus \ 3 & 3 \end{array}$

$= \dfrac{14}{x^2+3x-10} + \dfrac{3(x+3)}{(x+2)(x+5)}$

$\begin{array}{c|c} \otimes -10 & (5,-2) \\ \hline \oplus \ 3 & \end{array}$

$$= \frac{14}{(x-2)(x+5)} \cdot \frac{(x+2)}{\cdot (x+2)} + \frac{3(x+3)}{(x+2)(x+5)} \cdot \frac{(x-2)}{\cdot (x-2)}$$

$$= \frac{14(x+2) + 3(x+3)(x-2)}{(x-2)(x+2)(x+5)} = \frac{14x+28 + 3(x^2+x-6)}{(x-2)(x+2)(x+5)}$$

$$= \frac{14x+28 + 3x^2 + 3x - 18}{(x-2)(x+2)(x+5)} = \frac{3x^2 + 17x + 10}{(x-2)(x+2)(x+5)}$$

$$= \frac{(x+5)(3x+2)}{(x-2)(x+2)(x+5)}$$

$$= \boxed{\frac{3x+2}{(x-2)(x+2)}}$$

$\times$ 20 | (15,2)
$+$ 17 | 17

$3x^2 + 15x + 2x + 10$

$= 3x(x+5) + 2(x+5)$

$= (x+5)(3x+2)$

e) $\dfrac{\dfrac{1}{x-1} + \dfrac{2}{x+2}}{\dfrac{2}{x+2} - \dfrac{1}{x-3}}$ ← ÷
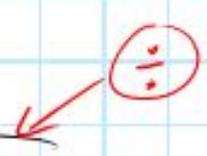
must be one fraction
before flip

$$= \left(\frac{1}{x-1} \cdot \frac{(x+2)}{(x+2)} + \frac{2}{x+2} \cdot \frac{(x+1)}{(x-1)} \right) \div \left(\frac{2}{x+2} \cdot \frac{(x-3)}{(x-3)} - \frac{1}{x-3} \cdot \frac{(x+2)}{(x+2)} \right)$$

$$= \left(\frac{x+2 + 2(x-1)}{(x-1)(x+2)} \right) \div \left(\frac{2(x-3) - (x+2)}{(x+2)(x-3)} \right)$$

$$= \left(\frac{3x}{(x-1)(x+2)} \right) \div \left(\frac{x-8}{(x+2)(x-3)} \right)$$

$$= \frac{3x}{(x-1)(x+2)} \cdot \frac{(x+2)(x-3)}{x-8} = \boxed{\frac{3x(x-3)}{(x-1)(x-8)}}$$

Rational Equations

① State Restrictions

② Multiply both sides to clear denominators

③ Solve (Exclude restrictions!)

Ex   Solve

a) $\left(\dfrac{6}{x-4} = \dfrac{5x}{x-4} + x\right)\cdot(x-4)$ $\underline{x \neq 4}$

$6 = 5x + x(x-4)$

$6 = 5x + x^2 - 4x$

$0 = x^2 + x - 6$

$\boxed{\begin{array}{c|c} \times & -6 \\ \hline + & 1 \end{array}}\ (3,-2)$

$0 = (x+3)(x-2)$

$x+3=0 \qquad x-2=0$

$\Rightarrow x=-3 \qquad x=2$

$\boxed{x = -3, 2}$

check: $x=-3$ $\qquad\qquad\qquad\qquad$ $x=2$

$\dfrac{6}{-3-4} = \dfrac{5(-3)}{-3-4} + (-3)$ $\qquad$ $\dfrac{6}{2-4} = \dfrac{5(2)}{2-4} + 2$

$\dfrac{6}{-7} = \dfrac{-15}{-7} - 3$ $\qquad\qquad$ $\dfrac{6}{-2} = \dfrac{10}{-2} + 2$

$-\dfrac{6}{7} = \dfrac{15}{7} - \dfrac{21}{7}$ $\qquad\qquad$ $-3 = -5 + 2$

$-\dfrac{6}{7} = -\dfrac{6}{7}\ \checkmark$ $\qquad\qquad\qquad$ $-3 = -3\ \checkmark$

b) $\dfrac{x}{x-5} - \dfrac{3}{x+1} = \dfrac{30}{x^2-4x-5}$ $\begin{array}{c} \times -5 \\ + -4 \end{array} \; (-5, 1)$

$(x+1)(x-5) \left(\dfrac{x}{x-5} - \dfrac{3}{x+1} = \dfrac{30}{(x-5)(x+1)} \right)$ $x \neq 5, -1$

$$x(x+1) - 3(x-5) = 30$$

$$x^2 + x - 3x + 15 = 30$$

$$x^2 - 2x - 15 = 0 \qquad \begin{array}{c} \times -15 \\ + -2 \end{array} \; (-5, 3)$$

$$(x-5)(x+3) = 0$$

$\cancel{x=5} \qquad x=-3$

$\boxed{x = -3}$

Weird Cases ("where did the X go?")

c) $\left(\dfrac{4}{2x-1} = \dfrac{2}{x+3} \right) \cdot (2x-1)(x+3)$ $x \neq -3, \dfrac{1}{2}$

$$4(x+3) = 2(2x-1)$$

$$4x + 12 = 4x - 2$$

when is $4 = 0$?

$$4(x+3) = 2(2x-1)$$

$$4x + 12 = 4x - 2$$

$$14 = 0 \quad \textcircled{?} \quad \text{when is } 14 = 0?$$
$$\text{Never!}$$

NO answer

d) $\dfrac{5}{x-7} - \dfrac{1}{2x} = \dfrac{9x+7}{2x^2-14x}$ $x \neq 0, 7$

$$\left(\dfrac{5}{x-7} - \dfrac{1}{2x} = \dfrac{9x+7}{2x(x-7)} \right) \cdot (x-7)(2x)$$

$$5(2x) - (x-7) = 9x + 7$$

$$10x - x + 7 = 9x + 7$$

$$0 = 0 \qquad \text{when is } 0 = 0?$$
$$\text{Always!}$$

$$\boxed{x \in \mathbb{R}, \ x \neq 0, 7}$$

Graphing Rational Functions

Rational Function: $\dfrac{g(x)}{h(x)}$

Domain: all x is o.k.

Except $h(x) = 0$

[Restrictions]

When $h(x) = 0$, we generally see Vertical Asymptotes

$f(x) = \dfrac{1}{x}$

vertical Asymptote: $x = 0$

Horizontal Asymptote: $y = 0$

$f(x) = \dfrac{2-x}{x-1}$

vertical Asymptote: $x = 1$

Horizontal Asymptote: $y = -1$

VA: when denominator $= 0$ and $y \to \pm \infty$

$x = \underline{\quad}$

HA: what happens when $x \to \pm \infty$ (Far left/right)

To find it, take the ratio of the largest degree term on top and bottom, simplify, then plug in $x \to \infty$ $y = \underline{\quad}$

⊛ Not all rational functions have one or both types of Asymptotes

Ex   Find all asymptotes

a) $\dfrac{1}{x^2-4}$

$= \dfrac{1}{(x+2)(x-2)}$

VA: Denom $= 0$ $\boxed{x = 2, -2}$

HA: $\dfrac{1}{x^2}$, as $x \to \infty$

$\dfrac{1}{x^2} \to 0$

$x \to 2^+$ $y \to \infty$ $\boxed{y=0}$

$x \to 2^-$ $y \to -\infty$

$x \to -2^+$ ← on the right $y \to -\infty$

$x \to -2^-$ $y \to \infty$
   on the left

$x \to \infty$ $y \to 0^+$ ← just bigger

$x \to -\infty$ $y \to 0^+$

b) $\dfrac{2x^2}{x-3}$

VA: $\boxed{x = 3}$

HA: $\dfrac{2x^2}{x} = 2x$ as $x \to \infty$

$2x \to \infty$

$\boxed{\text{NO HA}}$

c) $\dfrac{5x^2}{2x^2+1}$ VA: $\boxed{\text{NO VA}}$

HA: $\dfrac{5x^2}{2x^2} = \dfrac{5}{2}$ as $x \to \infty$

$\boxed{y = \dfrac{5}{2}}$ $\dfrac{5}{2} \to \dfrac{5}{2}$

How to Graph Rational Functions

① Find all VA/HA and plot them

② Find all x/y intercepts and plot them.

③ Use a Calculator to see what happens near each asymptote (on each side)

④ Connect it all using smooth Curves

Ex   Graph. Label all intercepts/Asymptotes. State Domain/Range

a) $f(x) = \dfrac{2}{x^2 - x - 2}$ VA: $x = 2, -1$

HA: $\dfrac{2}{x^2}$ as $x \to \infty$

$\dfrac{\otimes -2 \;\big|\; (-2, 1)}{\oplus -1 \;\big|\;}$ $= \dfrac{2}{(x-2)(x+1)}$ $\dfrac{2}{x^2} \to 0$

$y = 0$

$\uparrow$ $\uparrow$

y-int: f(

$\otimes -2 \mid (-2,1)$

$\oplus -1$

$= \dfrac{2}{(x-2)(x+1)}$

$x=0$
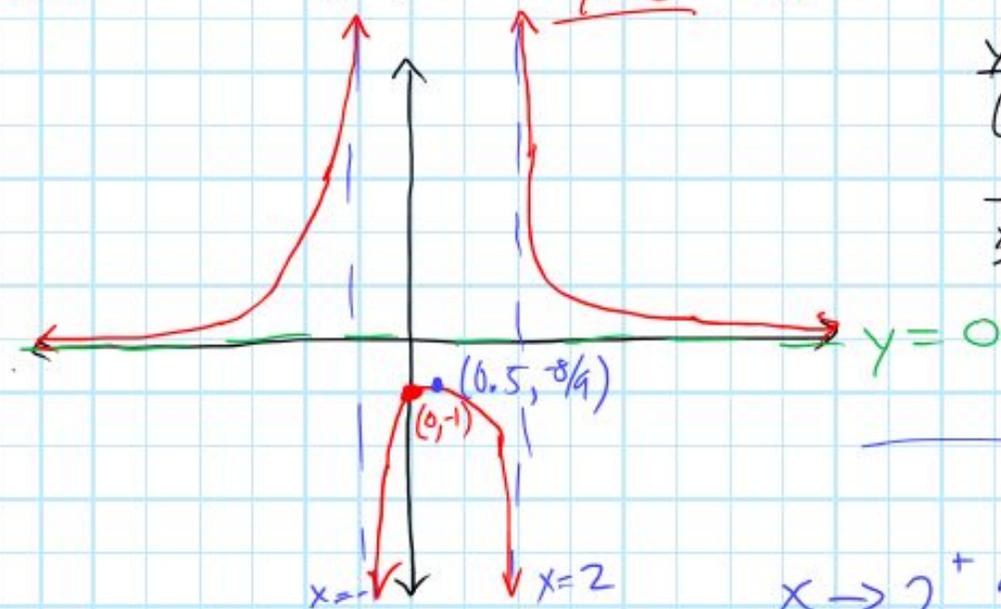
$\dfrac{2}{x^2} \to 0$

y-int: $f(0) = \dfrac{2}{(0)^2 - 0 - 2}$

$(0, -1)$

$= \dfrac{2}{-2} = -1$

x-int: $\left(\dfrac{2}{x^2 - x - 2} = 0 \right)(x^2 - x - 2)$

$2 = 0$ NO answer

no x-int

$y = 0$

$(0.5, 8/9)$

$(0, -1)$

$x = -1$ $x = 2$

$x \to 2^+$: $x = 2.01$ $y = \dfrac{2}{(2.01)^2 - (2.01) - 2} = 66.45$ $y \to \infty$

$x \to 2^-$: $x = 1.99$ $y = -66.89$ $y \to -\infty$

$x \to -1^+$: $x = -0.99$, $y = -66.89$ $y \to -\infty$

$x \to -1^-$: $x = -1.01$, $y = 66.45$ $y \to \infty$

$x \to \infty$: $x = 100$, $y = 0.0002$ $y \to 0^+$

$x \to -\infty$: $x = -100$, $y = 0.00019$ $y \to 0^+$

Domain: $x \in \mathbb{R}$ $x \neq -1, 2$

Range: $y > 0$ and $y < -\dfrac{8}{9}$

b)

$$f(x) = \frac{x^2 - 3x - 4}{x^2 + 2x}$$

$$= \frac{x^2 - 3x - 4}{x(x+2)}$$

VA: $\boxed{x = 0, -2}$

HA: $\frac{x^2}{x^2} = 1$ as $x \to \infty$

$1 \to 1$

$\boxed{y = 1}$
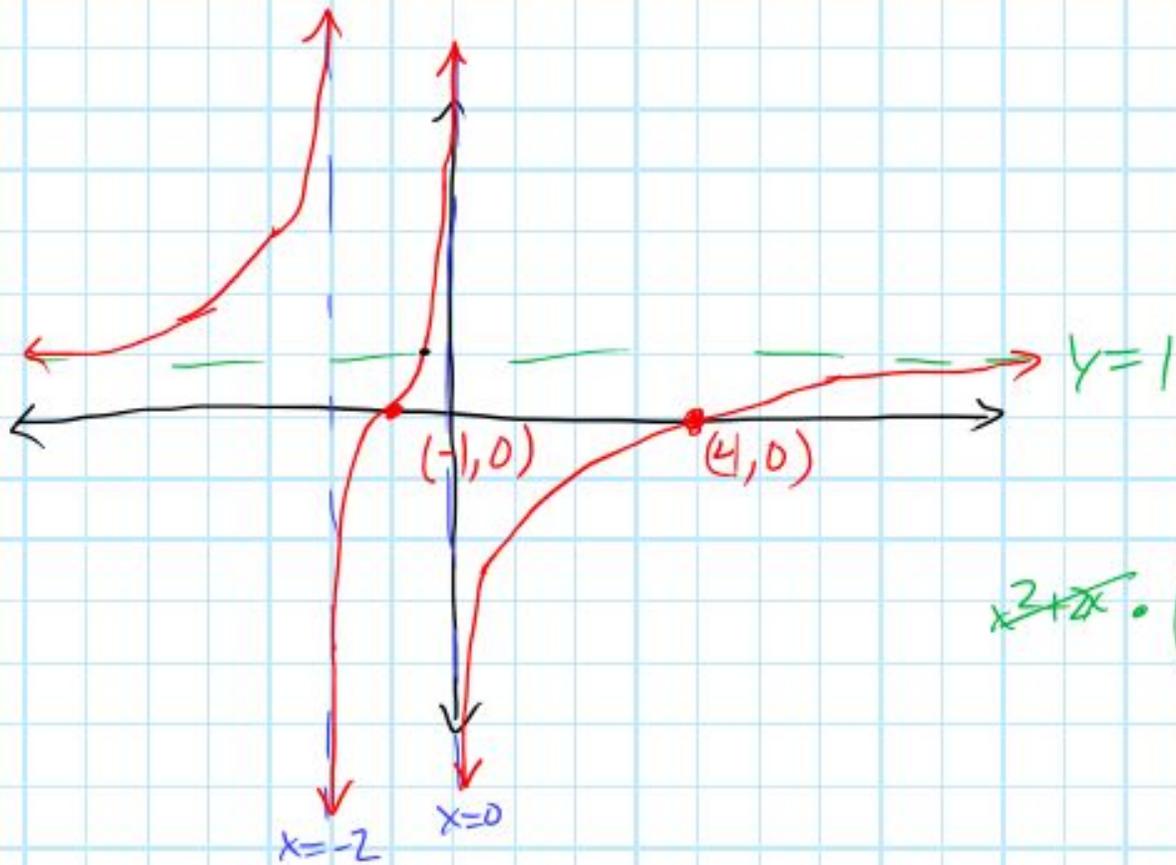
Y int: $= \frac{0^2 - 3 \cdot 0 - 4}{0^2 + 2 \cdot 0}$

$$= \frac{-4}{0}$$

$\underline{No\ yint}$

X int:

$x^2 + 2x \cdot \left(\frac{x^2 - 3x - 4}{x^2 + 2x} = 0 \right)$

$x^2 - 3x - 4 = 0$

$(x - 4)(x + 1) = 0$

$(4, 0), (-1, 0)$

$y = 1$

$(-1, 0)$ $(4, 0)$

$x = -2$ $x = 0$

$x \to 0^+, \ y \to -\infty$

$x \to 0^+$, $y \to -\infty$

$x = 0.01$, $y = -200$

$x \to 0^-$, $y \to \infty$

$x = -0.01$, $y = 199$

$x \to -2^+$, $y \to -\infty$

$x = -1.99$, $y = -298$

$x \to -2^-$, $y \to \infty$

$x = -2.01$ $y = 302$

$(x - 1)(x+1)$

$(4, 0)$, $(-1, 0)$

$x \to \infty$, $y \to 1^-$

$x = 100$, $y = 0.951$

$x \to -\infty$, $y \to 1^+$

$x = -100$ $y = 1.051$

Domain: $x \in \mathbb{R}$
$x \neq -2, 0$

Range: $y \in \mathbb{R}$

Ex 1 The sum of a number and 3 times its reciprocal is $\frac{28}{5}$. Find the number(s).

x – our number

$\left(x + \dfrac{3}{x} = \dfrac{28}{5} \right) \cdot 5x$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$5x^2 + 15 = 28x$

$x = \dfrac{-(-28) \pm \sqrt{(-28)^2 - 4(5)(15)}}{2(5)}$

$5x^2 - 28x + 15 = 0$

$x = \dfrac{28 \pm \sqrt{484}}{10} = \dfrac{28 \pm 22}{10} = \boxed{5, \dfrac{3}{5}}$

Ex 2 A hot water tap takes 25 minutes less time than a cold water tap to fill a bathtub. Together they fill the bathtub in 2 hours. Find how long each tap takes to fill the tub on its own. (Round to two decimal places)

	Hot	Cold	total
time	x	$x + 25_{min}$	$2\,hrs = 120\,min$
rate	$\dfrac{1}{x}$	$\dfrac{1}{x+25}$	$\dfrac{1}{120}$

$rate = \dfrac{amount}{time}$

me	other	together
10 min	5 min	?
$\dfrac{6\ tests}{1\ hr}$	$\dfrac{12\ tests}{1\ hr}$	$\dfrac{18\ tests}{1\ hr}$

$\left(\dfrac{1}{x} + \dfrac{1}{x+25} = \dfrac{1}{120} \right) \cdot x(x+25) \cdot 120$

$120(x+25) + 120x = x(x+25)$

$120x + 3000x + 120x = x^2 + 25x$

$0 = x^2 - 215x - 3000$

$x = \dfrac{-(-215) \pm \sqrt{(-215)^2 - 4(1)(-3000)}}{2(1)}$

$x = \dfrac{215 \pm \sqrt{58225}}{2}$

$x = 228.15, -13.15$

$\boxed{\text{Hot: } 228.15\,min \qquad \text{Cold: } 253.15\,min}$

Ex 3 A speedboat travels 45km downriver with the current, then turns around and travels upriver back to where it started. If the total trip time was 3.5 hours and the speed of the current is 12 km/h, what is the speed of the boat in still water? (Round to 2 decimal places)

$$D = s \cdot t \Rightarrow t = \frac{D}{s}$$

	Dist	Speed	time
Down River	$45\,km$	$x + 12 \frac{km}{h}$	$\dfrac{45}{x+12}$
UP River	$45\,km$	$x - 12 \frac{km}{h}$	$\dfrac{45}{x-12}$

total = 3.5 hrs

$$\left(\frac{45}{x+12} + \frac{45}{x-12} = 3.5 \right) \cdot (x+12)(x-12)$$

$$45(x-12) + 45(x+12) = 3.5(x+12)(x-12)$$

$$45x - 540 + 45x + 540 = 3.5(x^2 - 144)$$

$$0 = 3.5x^2 - 90x - 504$$

$$x = \frac{-(-90) \pm \sqrt{(90)^2 - 4(3.5)(-504)}}{2(3.5)} = \frac{90 \pm \sqrt{15156}}{7}$$

$$x = \boxed{30.44 \frac{km}{h}} \quad \cancel{-4.73}$$